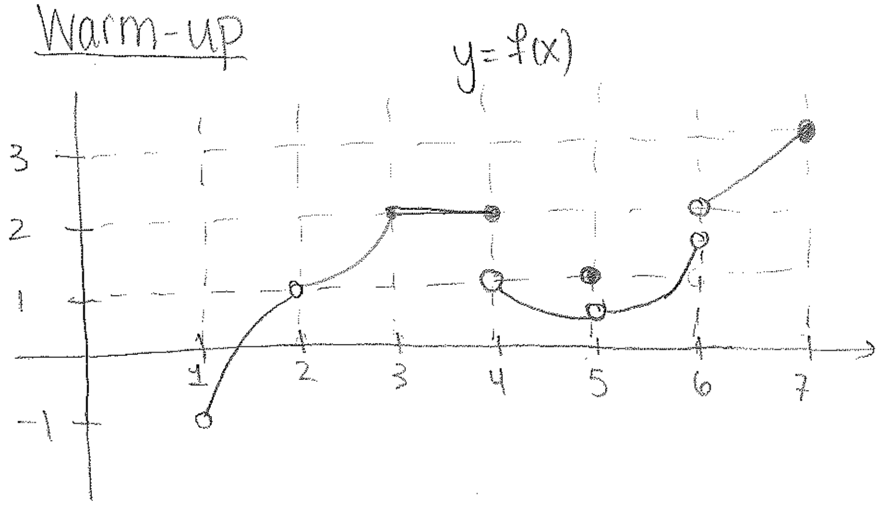
Jan. 14, 2014

Warm-up



1. Domain? $(1,2) \cup (2,6) \cup (6,7]$

2. Range? $(-1,3]$

3. For which values of a does $\lim\limits_{x \to a} f(x)$ not exist?

$\lim\limits_{x \to 1^-} f(x) = DNE$	$\lim\limits_{x \to 4^-} f(x) = 2$	$\lim\limits_{x \to 6^-} f(x) = 1.5$	$\lim\limits_{x \to 7^-} f(x) = 3$
$\lim\limits_{x \to 1^+} f(x) = -1$	$\lim\limits_{x \to 4^+} f(x) = 1$	$\lim\limits_{x \to 6^+} f(x) = 2$	$\lim\limits_{x \to 7^+} f(x) = DNE$

So for $x = -1, 4, 6, 7$

4. Which values a satisfy
$f(a)$ and $\lim\limits_{x \to a} f(x)$ exist, but $f(a) \neq \lim\limits_{x \to a} f(x)$

$a = 5$

Continuity

<u>DEF</u>: An interior pt of D is any pt <u>in</u> D which is not an endpt or an isolated pt.

In example above, everything in D except $x = 7$.

Let a be an interior pt of domain D of $f(x)$.

DEF: A function is continuous at a if $\lim_{x \to a} f(x) = a$. If it is not continuous at a, then f is discontinuous at a.

Determining Continuity checklist

☐ 1. Is a an interior pt? If no, stop. (we'll get back to this)

☐ 2. Do $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist?

☐ 3. Are they equal? (Does $\lim_{x \to a} f(x)$ exist?)

☐ 4. Does $f(a) = \lim_{x \to a} f(x)$?

If you check yes to all of the above, the function is continuous at a

Things we already know are continuous over their domains

- polynomials ($x^5 + x^3 + 2x^2 - 1 = f(x)$)
- rational functions ($f(x) = \dfrac{x^2 + 4x - 2}{x + 7}$ $D: (-\infty, -7) \cup (-7, \infty)$)
- trig functions
- exponential functions (a^x, e^x)
- absolute values
- inverse functions of all of these.

Example: Is $f(x) = \begin{cases} x^2 & x < 1 \\ x^3+2 & 1 \le x \end{cases}$ continuous?

Know continuous everywhere except, potentially, $x = 1$.

(1) ✓

(2) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^3 + 2 = 3$

Not equal

(3) Fails. $\lim_{x \to 1} f(x)$ DNE

What to do if "a" is an endpt:

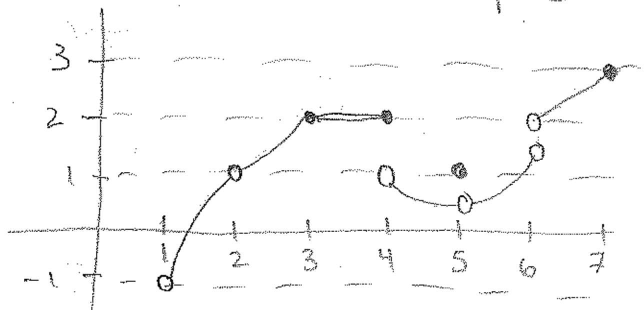
$f(x)$ is right cts at "a" if it is defined on int $[a, b)$ and $\lim_{x \to a^+} f(x) = f(a)$

this is just a # that makes sense

left cts at "a" if it is defined on int $(b, a]$ and $\lim_{x \to a^-} f(x) = f(a)$

cts = continuous

Go back to example:



What pts are not right cts? $x = 7, 5, 4$

What pts are not left cts? $x = 5$

If "a" is an endpt of D and $f(x)$ is left or right continuous at a, then we say $f(x)$ is continuous at a.

So now if "a" is endpt of D; just need to check

☐ $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ exists

☐ whichever limit exists must equal $f(a)$

DEF: A func is continuous over its domain D if
it is cts at every interior pt of D
and left/right cts at every endpt of D

(if no isolated pts in domain)

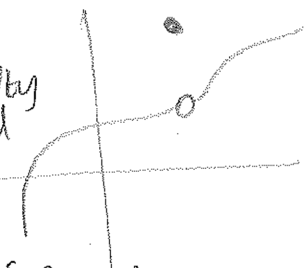
Fixing Discontinuities

Removable Discontinuity at a: *If f has a discontinuity at x=a and

"a" interior: $\lim_{x \to a} f(x) = L$ exists

"a" endpoint: $\lim_{x \to a^\pm} f(x) = L$ exists

these are true, it is called a removable discontinuity



can "Fix" this function: $\tilde{f}(x) = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases}$

Continuous Extension:

Let "a" be a hole in D

if $\lim_{x \to a} f(x) = L$ exists

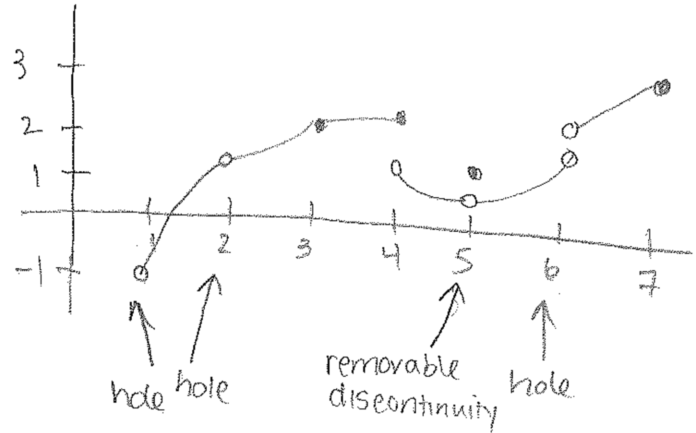
(or, if a would be an endpt, $\lim_{x \to a^\pm} f(x)$)

then f(x) has a cts extension

$$\bar{f}(x) = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases}$$

the same exact graph, but we extended the domain.

Back to example:



holes at

$x = 1, 2, 6$

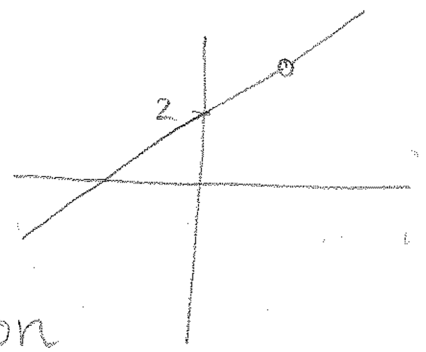
lim at 6 DNE

so only continuous extension at $x = 1, 2$

$$\bar{f}_1(x) = \begin{cases} f(x) & x \neq 1 \\ -1 & x = 1 \end{cases} \qquad \bar{f}_2(x) = \begin{cases} f(x) & x \neq 2 \\ 1 & x = 2 \end{cases}$$

__Ex 1__: $f(x) = \dfrac{x^2-4}{x-2}$ $\quad D: (-\infty, 2) \cup (2, \infty)$

$$= \frac{(x+2)(x-2)}{(x-2)} = (x+2)$$



hole at $x = 2$.

can find a continuous extension

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} x+2 = 4$$

$$\bar{f}(x) = \begin{cases} f(x) & x \neq 2 \\ 4 & x = 2 \end{cases} = x+2$$

__Ex 2__: $f(x) = \begin{cases} \sin x & x \neq \pi/3 \\ 0 & x = \pi/3 \end{cases}$ $\quad \lim_{x \to \pi/3} \sin x = \dfrac{\sqrt{3}}{2}$ $\quad \neq$

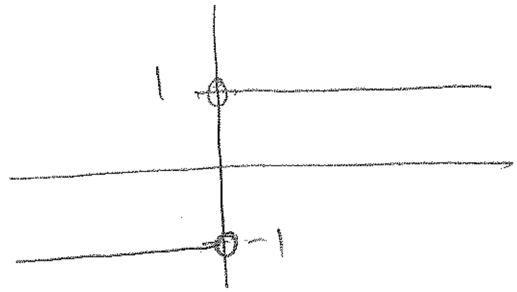
$f(\pi/3) = 0$

Removable discontinuity at $x = \pi/3$

we can fix it:

$\bar{f}(x) = \sin x$

Ex 3: $f(x) = \frac{|x|}{x}$



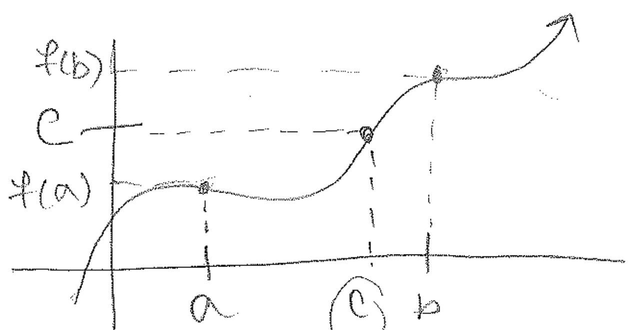
No continuous ext
OR removable
singularity

Why is continuity important?

Many reasons! One in particular:

The Intermediate Value Theorem

- f continuous on $[a, b]$
- Let C be a real # s.t. $f(a) < C < f(b)$ OR $f(b) < C < f(a)$
- there exists a real # c, such that $f(c) = C$



IVT tells us this value exists.

MORAL: graphs behave the way we want them to.

Ex: Show the equation $x^5 - 3x + 1 = 0$ has at least one solution in interval $[0,1]$

- We know $x^5 - 3x + 1 = f(x)$ is cts on $[0,1]$
- Let $C = 0$

$(0)^5 - 3(0) + 1 = 1$

$(1)^5 - 3(1) + 1 = -1$

so $f(0) > 0 > f(1)$

$\Rightarrow$ by IVT there is a solution